# Integrable Quasiclassical Deformations of Cubic Curves. *

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**Abstract**

A general scheme for determining and studying hydrodynamic type systems describing integrable deformations of algebraic curves is applied to cubic curves. Lagrange resolvents of the theory of cubic equations are used to derive and characterize these deformations.

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# 1 Introduction

The theory of algebraic curves is a fundamental ingredient in the analysis of integrable nonlinear differential equations as it is shown, for example, by its relevance in the description of the finite-gap solutions or the formulation of the Whitham averaging method [1]-[8]. A particularly interesting problem is characterizing and classifying integrable deformations of algebraic curves. In [6]-[7] Krichever formulated a general theory of dispersionless hierarchies of integrable models arising in the Whitham averaging method. It turns out that the algebraic orbits of the genus-zero Whitham equations determine infinite families of integrable deformations of a particular class of algebraic curves. A different approach for determining integrable deformations of general algebraic curves $\mathcal{C}$ defined by monic polynomial equations

$$\mathcal{C}: \quad F(p,k) := p^N - \sum_{n=1}^{N} u_n(k) p^{N-n} = 0, \quad u_n \in \mathbb{C}[k], \tag{1}$$

was proposed in [9]-[10]. It applies for finding deformations $\mathcal{C}(x,t)$ of $\mathcal{C}$ with the deformation parameters $(x,t)$, such that the multiple-valued function $\boldsymbol{p} = \boldsymbol{p}(k)$ determined by (1) obeys an equation of the form of conservation laws

$$\partial_t \boldsymbol{p} = \partial_x \boldsymbol{Q}, \tag{2}$$

where the flux $\boldsymbol{Q}$ is given by an element from $\mathbb{C}[k,p]/\mathcal{C}$,

$$\boldsymbol{Q} = \sum_{r=1}^{N} a_r(k,x,t)\boldsymbol{p}^{r-1}, \quad a_r \in \mathbb{C}[k].$$

Starting with (2), changing to the dynamical variables $u_n$ and using Lenard-type relations (see [10]) one gets a scheme for finding consistent deformations of (1). One should also note that (2) provides an infinite number of conservation laws, when one expands $\boldsymbol{p}$ and $\boldsymbol{Q}$ in Laurent series in $z$ with $k = z^r$ for some $r$. In this sense, we say that equation (2) is integrable.

Our strategy can be applied to the *generic case* where the coefficients (*potentials*) $u_n$ of (1) are general polynomials in $k$

$$u_n(k) = \sum_{i=0}^{d_n} u_{n,i} k^i,$$

with all the coefficients $u_{n,i}$ being considered as independent dynamical variables, i.e. $u_{n,i} = u_{n,i}(x,t)$. However, with appropriate modifications, the scheme can be also applied to cases in which constraints on the potentials are imposed. A complete description of these deformations for the generic case of hyperelliptic curves ($N = 2$) was given in [10].

The present paper is devoted to the deformations of cubic curves ($N = 3$)

$$p^3 - w\,p^2 - v\,p - u = 0, \quad u, v, w \in \mathbb{C}[k]\,, \tag{3}$$

and it considers not only the generic case but also the important constrained case $w \equiv 0$. Although some of the curves may be conformally equivalent (with for example the dispersionless Miura transformation), we will not discuss the classification problem under this equivalence in this paper (we will discuss the details of the problem elsewhere). In section 2 a general approach to construct integrable deformations of algebraic curves is reported briefly. Section 3 is devoted to the analysis of the cubic case (3). We emphasize the role of Lagrange resolvents, describe the Hamiltonian structure of integrable deformations and present several illustrative examples including Whitham type deformations.

# 2 Schemes of deformations of algebraic curves

In order to describe deformations of the curve $\mathcal{C}$ defined by (1), one may use the potentials $u_n$, as well as the $N$ branches $p_i = p_i(k)$ ($i = 1, \dots, N$) of the multiple-valued function function $\boldsymbol{p} = \boldsymbol{p}(k)$ satisfying

$$F(p,k) = \prod_{i=1}^{N} (p - p_i(k)). \tag{4}$$

The potentials can be expressed as elementary symmetric polynomials $s_n$ [11]-[13] of the branches $p_i$

$$u_n = (-1)^{n-1} s_n(p_1, p_2, \dots) = (-1)^{n-1} \sum_{1 \le i_1 < \dots < i_n \le N} p_{i_1} \cdots p_{i_n}. \tag{5}$$

However, notice that, according to the famous Abel theorem [11], for $N > 4$ the branches $p_i$ of the generic equation (1) cannot be written in terms of the potentials $u_n$ by means of rational operations and radicals.

There is an important result concerning the branches $p_i$ which is useful in our analysis. Let $\mathbb{C}((\lambda))$ denote the field of Laurent series in $\lambda$ with at most a finite number of terms with positive powers

$$\sum_{n=-\infty}^{N} c_n \lambda^n, \quad N \in \mathbb{Z}.$$

Then we have [14, 15] :

**Theorem 1.** *(***Newton Theorem***)*

*There exists a positive integer $l$ such that the $N$ branches*

$$p_i(z) := \Big(p_i(k)\Big)\Big|_{k=z^l}, \tag{6}$$

*are elements of $\mathbb{C}((z))$. Furthermore, if $F(p,k)$ is irreducible as a polynomial over the field $\mathbb{C}((k))$ then $l_0 = N$ is the least permissible $l$ and the branches $p_i(z)$ can be labelled so that*

$$p_i(z) = p_N(\epsilon^i z), \quad \epsilon := \exp\frac{2\pi i}{N}.$$

**Notation convention** *Henceforth, given an algebraic curve $\mathcal{C}$ we will denote by $z$ the variable associated with the least positive integer $l_0$ for which the substitution $k = z^{l_0}$ implies $p_i \in \mathbb{C}((z))$, $\forall i$. The number $l_0$ will be referred to as the Newton exponent of the curve .*

For the generic case the method proposed in [10] may be summarized as follows : Given an algebraic curve (1), we define an evolution equation of the form

$$\partial_t \mathbf{u} = J_0\Big(T\,\nabla_{\mathbf{u}} R\Big)_+, \quad R(z,\boldsymbol{p}) = \sum_i f_i(z)\,p_i, \tag{7}$$

where $(\,\cdot\,)_+$ indicates the part of non-negative powers of a Laurent series in $k$ and

$$f_i \in \mathbb{C}((z)), \quad \nabla_{\mathbf{u}} R := \Big(\frac{\partial R}{\partial u_1}\cdots\frac{\partial R}{\partial u_N}\Big)^\top,$$

$$J_0 := T^\top V^\top \partial_x V, \tag{8}$$

$$T := \begin{pmatrix} 1 & -u_1 & \cdots & -u_{N-1} \\ 0 & 1 & \cdots & -u_{N-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix}, \quad V := \begin{pmatrix} 1 & p_1 & \cdots & p_1^{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & p_N & \cdots & p_N^{N-1} \end{pmatrix}. \tag{9}$$

Let $d_{nm}$ and $d_n$ be the degrees of the matrix elements $(J_0)_{nm}$ and the potentials $u_n$ as polynomials in $k$, respectively. Then (7) defines a deformation of the curve, if $d_{nm}$ and $d_n$ satisfy the consistency conditions

$$\max\{d_{nm}, m = 1, 2, 3\} \leq d_n + 1, \quad n = 1, 2, 3, \tag{10}$$

and the components of $\nabla_{\mathbf{u}} R$ are in $\mathbb{C}((k))$ with $k = z^{l_0}$.

Equivalently, in terms of branches

$$\mathbf{p} := (p_1, \dots, p_N)^\top,$$

the system (7) can be written as

$$\partial_t \, \mathbf{p} = \partial_x (V \, \mathbf{r}_+), \tag{11}$$

where

$$\mathbf{r} := T \, \nabla_{\mathbf{u}} R(z, \boldsymbol{p}) = V^{-1} \mathbf{f}(z), \tag{12}$$

with $\mathbf{f}(z) := (f_1(z), \dots, f_N(z))^\top$. Notice that $\mathbf{r}$ is a solution of the Lenard relation

$$J_0 \mathbf{r} = 0. \tag{13}$$

Although there is not a general procedure for analyzing constrained cases, one may try a similar strategy. Firstly, we start from the equation for branches (2) and then, by expressing the potentials in terms of the independent branches only, we look for a formulation of the flows as

$$\partial_t \mathbf{u} = J_0 \mathbf{a}, \quad \mathbf{a} := (a_1, \dots, a_N)^\top, \tag{14}$$

for a certain operator $J_0$. Finally, we use solutions $\mathbf{r}$ of Lenard relations (13) and set $\mathbf{a} = \mathbf{r}_+$.

Another scheme for defining integrable deformations of algebraic curves of genus *zero* (i.e. rational curve) is implicit in the theory of integrable systems of dispersionless type developed in [7, 8], which we refer to as the Whitham deformations. It concerns with algebraic curves characterized by equations of the form

$$k = p^N + v_{N-2} p^{N-2} + \dots + v_0 + \sum_{r=1}^{M} \sum_{i=1}^{n_r} \frac{v_{r,i}}{(p - w_r)^i}, \tag{15}$$

where $v_n$, $v_{r,i}$, $w_r$ are $k$-independent coefficients. These curves arise in the theory of algebraic orbits of the genus-zero Whitham hierarchy [7, 8], where the function $k$ represents the Landau-Ginzburg potential of the associated topological field theory. We may rewrite the equation of the curve (15) in the polynomial form (1) with potentials $u_n$ of degrees $d_n \leq 1$ and satisfying a certain system of constraints.

To describe the deformations of (15) determined by Whitham flows we introduce local coordinates $\{z_0, z_1, \ldots, z_M\}$ of the extended $p$-plane at the punctures $\{w_0 := \infty, w_1, \ldots, w_M\}$ such that

$$k = z_0^N = z_1^{n_1} = \ldots = z_{n_M}^{n_M}. \tag{16}$$

It is clear that there are $N$ branches of $\boldsymbol{p}$ which have expansions in powers of $k^{1/N}$ and that, for each puncture $w_r$, $(r = 1, \ldots, M)$, there are $n_r$ branches of $\boldsymbol{p}$ having expansions in powers of $k^{1/n_r}$. Therefore, the Newton exponent $l_0$ is given by the least common multiple of the set of integers $\{N, n_1, \ldots, n_M\}$. Furthermore, it is clear that only in the absence of finite punctures $(M = 0)$ the curve (15) is irreducible over $\mathbb{C}((k))$.

At each puncture in $\{\infty, w_1, \ldots, w_M\}$, there is an infinite family of Whitham deformations of (15). They can be expressed by equations of the form (see [7, 8])

$$\partial_t \boldsymbol{p} = \partial_x \boldsymbol{Q}_{\alpha,n}, \tag{17}$$

where

$$\begin{cases} \boldsymbol{Q}_{\alpha,n} = & (z_\alpha^n)_\oplus(\boldsymbol{p}), \quad \alpha = 0, 1, \ldots, M, \ n \geq 1 \\ \\ \boldsymbol{Q}_{r,0} = & \ln(\boldsymbol{p} - w_r), \quad r = 1, \ldots, M. \end{cases}$$

Here $(z_\alpha^n)_\oplus$ stands for the singular part of $z_\alpha^n(p)$ at the puncture $w_\alpha$, with $(z_r^n)_\oplus(\infty) = 0$ for $1 \leq r \leq M$. There exist also commuting flows for the negative $n$ in (17) with logarithmic terms which correspond to the descendant flows of $Q_{r,0}$ (see [8] for the details).

In the absence of finite punctures $(M = 0)$, Whitham deformations become the dispersionless Gelfand-Dikii flows. They can be described by our scheme [10] as the reductions $u_1 \equiv 0$, $u_N = k - v_0$ of the generic case corresponding to $d_n = \delta_{Nn}$. However, for $M \geq 1$ it can be seen that, in general, Whitham deformations of (15) are not reductions of the flows (7) provided by our method. Some examples of this situation for cubic curves are shown below.

## 3 Deformations of cubic curves

For our subsequent analysis we introduce a basic tool of the theory of third order polynomial equations [11]: the so called *Lagrange resolvents*, defined by

$$\mathcal{L}_i := \sum_{j=1}^{3} (\epsilon^i)^j \, p_j, \quad i = 1, 2, 3, \quad \epsilon := e^{\frac{2\pi\, i}{3}}, \tag{18}$$

or, equivalently,

$$\begin{cases} \mathcal{L}_1 : & = \epsilon\, p_1 + \epsilon^2\, p_2 + p_3, \\ \mathcal{L}_2 : & = \epsilon^2\, p_1 + \epsilon\, p_2 + p_3, \\ \mathcal{L}_3 : & = p_1 + p_2 + p_3, \end{cases}$$

They can be expressed in terms of the potentials $\mathbf{u} = (w, v, u)^\top$ by using the identities

$$\begin{cases} \mathcal{L}_1 \cdot \mathcal{L}_2 = 3\, v + w^2, \quad \mathcal{L}_3 = w, \\ \\ \mathcal{L}_1^3 + \mathcal{L}_2^3 = 27u + 9vw + 2\, w^3, \end{cases}$$

which lead to

$$\begin{cases} 2\, \mathcal{L}_1^3 = 27u + 9vw + 2\, w^3 \quad + \sqrt{(27u + 9\, vw + 2\, w^3)^2 - 4(3v + w^2)^3}, \\ \\ 2\, \mathcal{L}_2^3 = 27u + 9vw + 2\, w^3 \quad - \sqrt{(27u + 9\, vw + 2\, w^3)^2 - 4(3v + w^2)^3}. \end{cases}$$

The fundamental advantage of Lagrange resolvents is that they provide explicit expressions of the branches $p_i$ in terms of the potentials according to Cardano formulas

$$3\, p_i = \sum_{j=1}^{3} (\epsilon^{-i})^j \, \mathcal{L}_j, \quad i = 1, 2, 3, \tag{19}$$

or, equivalently,

$$\begin{cases} 3\, p_1 & = \epsilon^2\, \mathcal{L}_1 + \epsilon\, \mathcal{L}_2 + \mathcal{L}_3, \\ 3\, p_2 & = \epsilon\, \mathcal{L}_1 + \epsilon^2\, \mathcal{L}_2 + \mathcal{L}_3, \\ 3\, p_3 & = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3. \end{cases}$$

As we will prove below, the Lagrange resolvents are essential to determine consistent deformations of cubic equations.

### 3.1 Generic case

As it was found in [10], for $N = 3$ the operator $J_0$ reads

$$J_0 = \begin{pmatrix} 3\partial_x & w\partial_x + w_x & (2v + w^2)\partial_x + (2v + w^2)_x \\ -2w\,\partial_x & 2v\,\partial_x + v_x & (3u + vw)\partial_x + 2u_x + 2vw_x \\ -v\partial_x & 3u\partial_x + u_x & uw\partial_x + 2u\,w_x \end{pmatrix} \tag{20}$$

Thus if $d_1, d_2$ and $d_3$ are the degrees in $k$ of the potential functions $w$, $v$ and $u$, respectively, the consistency conditions (10) are

$$\begin{cases} d_1 \leq 1, \quad d_2 \leq d_1 + 1, \\ d_3 \leq d_2 + 1, \quad d_2 \leq d_3 + 1, \end{cases}$$

which lead to the following twelve nontrivial choices for $(d_1, d_2, d_3)$

$$\begin{array}{l} (0,0,1),\ (0,1,0),\ (0,1,1),\ (0,1,2), \\ (1,0,0),\ (1,0,1),\ (1,1,0),\ (1,1,1), \\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,3). \end{array} \tag{21}$$

By using (19)-(20) it is straightforward to determine the Newton exponent $l_0$ for each of the cases (21). Thus one finds three categories

| $l_0$ | 3 | 2 | 1 |
|---|---|---|---|
| | (0,0,1), (0,1,2) | (0,1,0), (0,1,1)<br>(1,0,0), (1,1,2) | (1,0,1), (1,1,0)<br>(1,1,1), (1,2,1)<br>(1,2,2), (1,2,3) |

Only the cases with $l_0 = 3$ correspond to irreducible curves over the field $\mathbb{C}((k))$. We also note here that our deformations for the trigonal curves (1) in the generic case allow one to have only the curves with genus less than or equal to one (the details will be discussed elsewhere).

Once the Newton exponent $l_0$ is known, in order to derive the associated hierarchy of integrable deformations according to our scheme, two steps are still required:

1. To determine the functions $R(z,\boldsymbol{p})=\sum_i f_i(z)\,p_i$ such that the components of $\nabla_{\mathbf{u}}R$ are in $\mathbb{C}((k))$ with $k=z^{l_0}$.

2. To find the explicit form of the gradients $\nabla_{\mathbf{u}}R$ in terms of the potentials.

Both problems admit a convenient treatment in terms of Lagrange resolvents. Thus by introducing the following element $\sigma_0$ of the Galois group of the curve

$$\sigma_0(p_i)(z):=p_i(\epsilon_0\,z),\quad \epsilon_0:=e^{\frac{2\pi i}{l_0}}, \tag{22}$$

we see that our first problem can be fixed by determining functions $R$ invariants under $\sigma_0$ i.e. $R(\epsilon_0\,z,\sigma_0\,\boldsymbol{p})=R(z,\boldsymbol{p})$. In this way, we have the following forms of $R$:

For the case $l_0=3$, the element $\sigma_0$ is given by the permutation

$$\sigma_0=\begin{pmatrix} p_1 & p_2 & p_3\\ p_2 & p_3 & p_1\end{pmatrix}, \tag{23}$$

or, in terms of Lagrange resolvents,

$$\sigma_0=\begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3\\ \epsilon^2\mathcal{L}_1 & \epsilon\mathcal{L}_2 & \mathcal{L}_3\end{pmatrix}. \tag{24}$$

Thus we get the invariant functions

$$R=zf_1(z^3)\,\mathcal{L}_1+z^2f_2(z^3)\,\mathcal{L}_2+f_3(z^3)\,\mathcal{L}_3, \tag{25}$$

with $f_i(z^3)$ being arbitrary functions in $\mathbb{C}((z^3))$.

For the case $l_0=2$, $\sigma_0^2$ is the identity permutation, so that under the action of $\sigma_0$ two branches are interchanged while the other remains invariant. If we label the branches in such a way that

$$\sigma_0=\begin{pmatrix} p_1 & p_2 & p_3\\ p_2 & p_1 & p_3\end{pmatrix}, \tag{26}$$

then

$$\sigma_0=\begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3\\ \mathcal{L}_2 & \mathcal{L}_1 & \mathcal{L}_3\end{pmatrix}, \tag{27}$$

and we obtain the invariant functions

$$R=f_1(z^2)\Big(\mathcal{L}_1+\mathcal{L}_2\Big)+zf_2(z^2)\Big(\mathcal{L}_1-\mathcal{L}_2\Big)+f_3(z^2)\,\mathcal{L}_3, \tag{28}$$

where $f_i(z^2)$ are arbitrary functions in $\mathbb{C}((z^2))$.

For the case $l_0 = 1$, we have $z = k$ and $\sigma_0$ is the identity, so that any function $R(k, \boldsymbol{p})$ is invariant under $\sigma_0$.

Now the problem of finding the gradients of $R$ reduces to determine the gradients of the Lagrange resolvents. To this end we differentiate (19) and obtain

$$\begin{cases} \mathcal{L}_2 \nabla_{\mathbf{u}} \mathcal{L}_1 + \mathcal{L}_1 \nabla_{\mathbf{u}} \mathcal{L}_2 & = (2w\,, 3\,, 0)^\top, \\ \mathcal{L}_1^2 \nabla_{\mathbf{u}} \mathcal{L}_1 + \mathcal{L}_2^2 \nabla_{\mathbf{u}} \mathcal{L}_2 & = (2w^2 + 3v\,, 3w\,, 9)^\top, \end{cases}$$

so that

$$\begin{cases} (\mathcal{L}_1^3 - \mathcal{L}_2^3) \nabla_{\mathbf{u}} \mathcal{L}_1 & = \Big( (2w^2 + 3v)\mathcal{L}_1 - 2w\mathcal{L}_2^2\,, 3(w\mathcal{L}_1 - \mathcal{L}_2^2)\,, 9\mathcal{L}_1 \Big)^\top, \\ (\mathcal{L}_2^3 - \mathcal{L}_1^3) \nabla_{\mathbf{u}} \mathcal{L}_2 & = \Big( (2w^2 + 3v)\mathcal{L}_2 - 2w\mathcal{L}_1^2\,, 3(w\mathcal{L}_2 - \mathcal{L}_1^2)\,, 9\mathcal{L}_2 \Big)^\top. \end{cases}$$

Hence the gradients of the generic density $R$ for (25) and (28) are given as follows:

For $l_0 = 3$, we have

$$\begin{aligned} \nabla_{\mathbf{u}} R = \; & \frac{z f_1(z^3)}{\mathcal{L}_1^3 - \mathcal{L}_2^3} \begin{pmatrix} (2w^2 + 3v)\mathcal{L}_1 - 2w\mathcal{L}_2^2 \\ 3(w\mathcal{L}_1 - \mathcal{L}_2^2) \\ 9\mathcal{L}_1 \end{pmatrix} \\ & - \frac{z^2 f_2(z^3)}{\mathcal{L}_1^3 - \mathcal{L}_2^3} \begin{pmatrix} (2w^2 + 3v)\mathcal{L}_2 - 2w\mathcal{L}_1^2 \\ 3(w\mathcal{L}_2 - \mathcal{L}_1^2) \\ 9\mathcal{L}_2 \end{pmatrix} + f_3(z^3) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

For $l_0 = 2$, we get

$$\nabla_{\mathbf{u}} R = \frac{f_1(z^2)}{\mathcal{L}_1^3 - \mathcal{L}_2^3} \begin{pmatrix} (2w^2+3v)(\mathcal{L}_1 - \mathcal{L}_2) + 2w(\mathcal{L}_1^2 - \mathcal{L}_2^2) \\ \\ 3(w\mathcal{L}_1 - \mathcal{L}_2^2) - 3(w\mathcal{L}_2 - \mathcal{L}_1^2) \\ \\ 9(\mathcal{L}_1 - \mathcal{L}_2) \end{pmatrix}$$

$$+ \frac{z f_2(z^2)}{\mathcal{L}_1^3 - \mathcal{L}_2^3} \begin{pmatrix} (2w^2+3v)(\mathcal{L}_1 + \mathcal{L}_2) - 2w(\mathcal{L}_1^2 + \mathcal{L}_2^2) \\ \\ 3(w\mathcal{L}_1 - \mathcal{L}_2^2) + 3(w\mathcal{L}_2 - \mathcal{L}_1^2) \\ \\ 9(\mathcal{L}_1 + \mathcal{L}_2) \end{pmatrix} + f_3(z^2) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

From these expressions and (24) and (27) it follows that the corresponding components of $\nabla_{\mathbf{u}} R$ are in $\mathbb{C}((k))$.

**Example 1:** The case $l_0 = 3$ with $(d_1, d_2, d_3) = (0, 0, 1)$. Taking into account (20) and (19) it is clear that there are two trivial equations corresponding to $w_0$ and $u_1$. Then, we take for the potentials

$$w = 1, \quad v = v_0(x,t), \quad u = k + u_0(x,t).$$

Thus, by using (25) with

$$f_1 \equiv f_3 \equiv 0, \quad f_2(z^3) = \frac{27(1-\sqrt{3}i)}{4} z^3$$

we obtain

$$\left\{ \begin{array}{rcl} v_{0\,t} & = & \dfrac{5}{3}\,(2 + 27\,u_0 + 9\,v_0)\,\,u_{0\,x} + \\ \\ & & \dfrac{5}{18}\left(7 + 54\,u_0 + 36\,v_0 + 27\,v_0^2\right)\,v_{0\,x}, \\ \\ u_{0\,t} & = & \dfrac{5}{18}\,\left(-1 - 54\,u_0 + 27\,{v_0}^2\right)\,u_{0\,x} + \\ \\ & & \dfrac{5}{9}\,v_0\,\,(2 + 27\,u_0 + 9\,v_0)\,\,v_{0\,x}. \end{array} \right.$$

It can be checked that this system corresponds to the one obtained by setting $M = 0$, $N = 3$ in (15), and $\alpha = 0$, $n = 5$ in (17).

**Example 2:** The case $l_0 = 2$ with $(d_1, d_2, d_3) = (0, 1, 0)$, $(l_0 = 2)$. From (20) and (19) we see that $v_{1\,t} = 0$. We then take

$$w = w_0(x,t), \quad v = k + v_0(x,t), \quad u = u_0(x,t),$$

and

$$f_1(z^2) = z^4, \quad f_2 \equiv f_3 \equiv 0.$$

Thus it follows

$$\left\{\begin{array}{lcl} w_{0\,t} &=& 4\,\left(w_0\,u_{0\,x} + v_0\,v_{0\,x} + u_0\,w_{0\,x}\right), \\ v_{0\,t} &=& -2\,\left(w_0^2\,u_{0\,x} - 2\,u_0\,v_{0\,x} + u_0\,w_0\,w_{0\,x}\right) + \\ && 2v_0\,\left(2\,u_{0\,x} - w_0\,v_{0\,x}\right), \\ u_{0\,t} &=& -2\,\left(v_0\,w_0\,u_{0\,x} + u_0\,\left(-2\,u_{0\,x} + w_0\,v_{0\,x} + v_0\,w_{0\,x}\right)\right). \end{array}\right.$$

It turns out that this system can also be found among the Whitham deformations, by setting $M = 1$, $N = 2$ in (15), and $\alpha = 0$, $n = 4$ in (17).

**Example 3:** The case $l_0 = 2$ with $(d_1, d_2, d_3) = (1, 0, 0)$. From (19) and (20) it is easy to see that $\left(\dfrac{u_0}{w_1}\right)_t = 0$. If we choose

$$\left\{\begin{array}{l} w(k,x,t) = w_1(x,t)k + w_0(x,t), \quad v(k,x,t) = v_0(x,t), \\ u(k,x,t) = w_1(x,t), \end{array}\right.$$

and set

$$f_1(z^2) = z^4, \quad f_2 \equiv f_3 \equiv 0,$$

in (28), then the following system arises

$$\left\{\begin{array}{lcl} w_{1\,t} &=& 2\,w_1^{-2}(w_1\,w_{0\,x}\,-\,w_0\,w_{1\,x}), \\ w_{0\,t} &=& 2w_1^{-3}\left(w_1\,\left(v_{0\,x} + w_0\,w_{0\,x}\right) - \left(2\,v_0 + w_0^2\right)\,w_{1\,x}\right), \\ v_{0\,t} &=& w_1^{-3}\left(-4\,w_1\,w_{1\,x}\,+\,2\,v_0\,\left(w_1\,w_{0\,x}\,-\,w_0\,w_{1\,x}\right)\right). \end{array}\right.$$

This is one of the flows in the dispersionless Dym hierarchy corresponding to the curve, $w_1 k = p - w_0 - v_0 p^{-1} - w_1 p^{-2}$. Also note that the linear flow, i.e. $w_{1t} = c w_{1x}$ etc with $c =$constant, can be obtained by the choice $f_2 \propto z^{-2}$ with $f_1 = f_3 = 0$.

**Example 4:** The case $l_0 = 1$ with $(d_1, d_2, d_3) = (1, 0, 1)$. From (19) and (20) one finds that $\left(\dfrac{u_1}{w_1}\right)_t = 0$. By setting

$$\begin{cases} w(k,x,t) = w_1(x,t)k + w_0(x,t), \quad v(k,x,t) = v_0(x,t), \\ u(k,x,t) = w_1(x,t)k + u_0(x,t), \end{cases}$$

and

$$R = \frac{2(1+\sqrt{3}i)}{\sqrt{3}} k \mathcal{L}_1,$$

we obtain

$$\begin{cases} w_{1\,t} &= u_{0\,x} + w_{0\,x}, \\ w_{0\,t} &= w_1^{-2}\Big(w_1\ (v_{0\,x} + u_0\, w_{0\,x}) - (3+v_0)\ w_{1\,x} - w_0^2\, w_{1\,x} \\ & \quad +w_0\ (w_1\ (u_{0\,x} + 2\, w_{0\,x}) - u_0\, w_{1\,x})\Big), \\ v_{0\,t} &= w_1^{-2}\Big(w_1\ (2\ (2+v_0)\ u_{0\,x} + u_0\, v_{0\,x} + w_0\, v_{0\,x} + 2\, v_0\, w_{0\,x}) \\ & \quad -2\ (u_0\ (3+v_0) - (1-v_0)\ w_0)\ w_{1\,x}\Big), \\ u_{0\,t} &= w_1^{-2}\Big( - w_0\, w_1\, u_{0\,x} - 3\, u_0^2\, w_{1\,x} + v_0\ (w_1\, v_{0\,x} + (1-v_0)\ w_{1\,x}) \\ & \quad +u_0\ (w_1\ (4\, u_{0\,x} + w_{0\,x}) + w_0\, w_{1\,x})\Big). \end{cases}$$

We also note that the linear flow is obtained by choosing $R \propto \mathcal{L}_1$, and the higher flows in the hierarchy can be obtained by $R \propto k^n \mathcal{L}_1$.

**Example 5:** The case $l_0 = 1$ with $(d_1, d_2, d_3) = (1, 1, 0)$. From (19) and (20) we deduce that $\left(\dfrac{v_1}{w_1}\right)_t = 0$. Now we take

$$\begin{cases} w(k,x,t) = w_1(x,t)k + w_0(x,t), \quad v(k,x,t) = w_1(x,t)k + v_0(x,t), \\ u(k,x,t) = u_0(x,t) \end{cases}$$

and set

$$R = \frac{\sqrt{3}+i}{2\sqrt{3}} k \mathcal{L}_2 \,.$$

Then the following system is obtained

$$\begin{cases} w_{1\,t} &=& 2\,u_{0\,x} - v_{0\,x}, \\ w_{0\,t} &=& w_1^{-2}\Big(w_1\ ((3+2\,w_0)\ u_{0\,x} - (2+w_0)\ v_{0\,x} + (2\,u_0 - v_0)\ w_{0\,x}) \\ && +\,(v_0\ (2+w_0) - u_0\ (3+2\,w_0))\ w_{1\,x}\Big), \\ v_{0\,t} &=& w_1^{-2}\Big(w_1\ ((-2+4\,v_0 - 2\,w_0)\ u_{0\,x} + (2\,u_0 - 3\,v_0 + w_0)\ v_{0\,x}) \\ && +\,(v_0\ (2\,v_0 - w_0) + u_0\ (3 - 4\,v_0 + 2\,w_0))\ w_{1\,x}\Big), \\ u_{0\,t} &=& w_1^{-2}\Big(\ (-2\,v_0 + w_0)\ w_1\,u_{0\,x} - 6\,u_0^2\,w_{1\,x} \\ && +u_0\ (w_1\ (8\,u_{0\,x} - 3\,v_{0\,x} + w_{0\,x}) + 2\ (2\,v_0 - w_0)\ w_{1\,x})\Big). \end{cases}$$

## 3.2 Hamiltonian structures

The general structure of integrable deformations (7) does not exhibit a direct hamiltonian form. However, the analysis of particular cases reveals the presence of certain Hamiltonian structures. We look for a Hamiltonian operator $J$ such that for certain appropriate densities $R$ it verifies

$$J_0\Big(T\,\nabla_{\mathbf{u}} R\Big)_+ = J\Big(\nabla_{\mathbf{u}} R\Big)_+, \tag{29}$$

where

$$T := \begin{pmatrix} 1 & -w & -v \\ 0 & 1 & -w \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, if (29) holds then the flows (7) can be written in the pre-Hamiltonian form

$$\partial_t \mathbf{u} = J\Big(\nabla_{\mathbf{u}} R\Big)_+. \tag{30}$$

To achieve our aim we require a $k$-independent operator $T_0$ verifying

$$T\,\nabla_{\mathbf{u}} R = T_0\,\nabla_{\mathbf{u}} R, \tag{31}$$

so that $J := J_0 \cdot T_0$ is a Hamiltonian operator.

Let us consider first the case $l_0 = 3$. It involves two classes of cubic curves:

For the case with $(d_1, d_2, d_3) = (0, 0, 1)$, the potentials are of the form

$$w = w_0(x), \quad v(x) = v_0(x), \quad u = u_0(x) + k\,u_1(x).$$

The matrix $T$ is $k$-independent so that by setting $J = J_0 \cdot T$ we find the Hamiltonian operator

$$J = \begin{pmatrix} 3\partial_x & -2\partial_x \cdot w & -\partial_x \cdot v \\ \\ -2w\,\partial_x & 2w\partial_x \cdot w + 2v\partial_x + v_x & (3u + vw)\partial_x + 2u_x + wv_x \\ \\ -v\partial_x & (3u + vw)\partial_x + vw_x + u_x & v\partial_x \cdot v - 2uw\partial_x - (uw)_x \end{pmatrix}. \tag{32}$$

It represents the dispersionless limit of the Hamiltonian structure of the Boussinesq hierarchy.

For the case with $(0, 1, 2)$ the potentials now are

$$\left\{ \begin{array}{l} w = w_0(x), \quad v(x) = v_0(x) + k\,v_1(x), \\ u = u_0(x) + k\,u_1(x) + k^2\,u_2(x). \end{array} \right.$$

From (29) one deduces

$$\left\{ \begin{array}{l} T\,\nabla_{\mathbf{u}} \mathcal{L}_i = T_0\,\nabla_{\mathbf{u}} \mathcal{L}_i, \quad i = 1, 2, \\ T\,\nabla_{\mathbf{u}} \mathcal{L}_3 = \mathcal{L}_3, \end{array} \right. \tag{33}$$

where $T_0$ is the $k$-independent matrix

$$T_0 = \begin{pmatrix} -2 & w & 0 \\ 0 & 1 & -w \\ 0 & 0 & 1 \end{pmatrix}. \tag{34}$$

Moreover $J := J_0 \cdot T_0$ takes the Hamiltonian form

$$J = \begin{pmatrix} -6\partial_x & 4\partial_x \cdot w & 2\partial_x \cdot v \\ \\ 4w\,\partial_x & -2w\partial_x \cdot w + 2v\partial_x + v_x & (3u - vw)\partial_x + 2u_x - wv_x \\ \\ 2v\partial_x & (3u - vw)\partial_x - vw_x + u_x & -2uw\partial_x - (uw)_x \end{pmatrix}. \tag{35}$$

Thus by setting

$$R = zf_1(z^3)\mathcal{L}_1 + z^2 f_2(z^3)\mathcal{L}_2,$$

equation (7) reduces to the form (30).

For the remaining cases of $l_0 = 2$ and $l_0 = 1$, the situation is as follows:

1. For the sets of degrees $(0,1,0)$ and $(0,1,1)$ for $l_0 = 2$, the identities (33) with the same operator (34) hold, so that by setting

$$R = f_1(z^2)\Big(\mathcal{L}_1 + \mathcal{L}_2\Big) + zf_2(z^2)\Big(\mathcal{L}_1 - \mathcal{L}_2\Big),$$

   equation (7) reduces to the form (30) with the Hamiltonian operator (35).

2. For the sets of degrees (two cases of $l_0 = 2$ and all the cases of $l_0 = 1$),

$$(1,0,0),\ (1,0,1),\ (1,1,0),\ (1,1,1)$$
$$(1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,3),$$

   there is no $k$-independent operator $T_0$ satisfying (32) for $\nabla_{\mathbf{u}}\mathcal{L}_i$, $(i = 1,2)$.

## 3.3 Deformations of cubic curves with $w = 0$

Deformations of cubic curves of the form

$$p^3 - v\,p - u = 0, \tag{36}$$

cannot be obtained simply by setting $w = 0$ in the above analysis. Indeed, as it is clear from the expression (20) for $J_0$, the constraint $w = 0$ does not constitutes a reduction of the flows (7). Therefore, we have to apply our deformation scheme to (36) directly.

In terms of the branches $p_i$ the condition $w = 0$ reads

$$p_1 + p_2 + p_3 = 0,$$

which is preserved by deformations

$$\partial_t p_i = \partial_x (a_1 + a_2\, p_i + a_3\, p_i^2) \tag{37}$$

satisfying

$$3a_1 = -(p_1^2 + p_2^2 + p_3^2)\, a_3. \tag{38}$$

By expressing the potentials as functions of the branches $p_1$ and $p_2$

$$v = p_1^2 + p_2^2 + p_1 p_2, \quad u = -(p_1^2 p_2 + p_1 p_2^2), \tag{39}$$

and using (37) and (38), we obtain

$$\partial_t \mathbf{u} = \mathcal{J}_0\, \mathbf{a}, \quad \mathbf{u} := (v \quad u)^\top, \quad \mathbf{a} := (a_1 \quad a_2)^\top, \tag{40}$$

where

$$\begin{aligned} \mathcal{J}_0 &= \begin{pmatrix} 2p_1 + p_2 & 2p_2 + p_1 \\ -2p_1 p_2 - p_2^2 & -2p_1 p_2 - p_1^2 \end{pmatrix} \partial_x \begin{pmatrix} p_1 & \frac{1}{3}p_1^2 - \frac{2}{3}(p_2^2 + p_1 p_2) \\ p_2 & \frac{1}{3}p_2^2 - \frac{2}{3}(p_1^2 + p_1 p_2) \end{pmatrix} \\ &= \begin{pmatrix} 2v\partial_x + v_x & 3u\partial_x + 2u_x \\ 3u\partial_x + u_x & \frac{1}{3}(2v^2\partial_x + 2v v_x) \end{pmatrix}. \end{aligned} \tag{41}$$

According to our strategy for finding consistent deformations, we use Lenard type relations

$$\mathcal{J}_0 \boldsymbol{R} = 0, \quad \boldsymbol{R} := (r_1 \quad r_2)^\top, \ \ r_i \in \mathbb{C}((k)),$$

to generate systems of the form

$$\mathbf{u}_t = \mathcal{J}_0 \mathbf{a}, \quad \mathbf{a} := \boldsymbol{R}_+. \tag{42}$$

Here $(\,\cdot\,)_+$ and $(\,\cdot\,)_-$ indicate the parts of non-negative and negative powers in $k$, respectively. Now from the identity

$$\mathcal{J}_0 \mathbf{a} = \mathcal{J}_0 \boldsymbol{R}_+ = -\mathcal{J}_0 \boldsymbol{R}_-,$$

it is clear that a sufficient condition for the consistency of (42) is that the degrees $d_2$ and $d_3$ of $v$ and $u$ as polynomials of $k$ satisfy

$$d_3 \leq d_2 + 1, \quad 2d_2 \leq d_3 + 1.$$

Hence only four nontrivial cases arise for $(d_2, d_3)$

$$(0,1), \quad (1,1), \quad (1,2), \quad (2,3). \tag{43}$$

We notice that they represent the dispersionless versions of the standard Boussinesq hierarchy and all three hidden hierarchies found by Antonowicz, Fordy and Liu for the third-order spectral problem [16].

Solutions of the Lenard relation can be generated by noticing that the operator $\mathcal{J}_0$ admits the factorization

$$\mathcal{J}_0 = U^\top \cdot \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \partial_x \cdot U, \tag{44}$$

where

$$U := \begin{pmatrix} 2p_1 + p_2 & -2p_1p_2 - p_2^2 \\ 2p_2 + p_1 & -2p_1p_2 - p_1^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial p_1} & \frac{\partial u}{\partial p_1} \\ \frac{\partial v}{\partial p_2} & \frac{\partial u}{\partial p_2} \end{pmatrix}. \tag{45}$$

This shows two things:

**i)** $\mathcal{J}_0$ is a Hamiltonian operator.

**ii)** The gradients $\nabla_{\mathbf{u}} p_i$ of the branches $p_1$ and $p_2$ solve the Lenard relations.

Thus our candidates to deformations are the equations of the form

$$\partial_t \mathbf{u} = \mathcal{J}_0 \Big(\nabla_{\mathbf{u}} R\Big)_+, \quad R(z, \boldsymbol{p}) = f_1(z)\, p_1 + f_2(z)\, p_2, \tag{46}$$

At this point one applies the same strategy as that used for the curves (3) in subsection 3.1. We first determine the Newton exponents of the four cases (43) which turn to be given by

| $l_0$ | 3 | 2 | 1 |
|---|---|---|---|
| | (0,1)<br>(1,2) | (1,1) | (2,3) |

Then, with the help of Lagrange resolvents, we characterize the functions $R(z, \boldsymbol{p})$ verifying $\nabla_{\mathbf{u}} R \in \mathbb{C}((k))$ with $k = z^{l_0}$. In summary, one finds

For the case $l_0 = 3$,

$$R = z f_1(z^3)\,\mathcal{L}_1 + z^2 f_2(z^3)\,\mathcal{L}_2, \quad k = z^3. \tag{47}$$

For the case $l_0 = 2$,

$$R = f_1(z^2)\Big(\mathcal{L}_1 + \mathcal{L}_2\Big) + z f_2(z^2)\Big(\mathcal{L}_1 - \mathcal{L}_2\Big), \quad k = z^2. \tag{48}$$

For the case $l_0 = 1$, we have $z = k$, so that any function $R(k, \boldsymbol{p}) = f_1(k)\mathcal{L}_1 + f_2(k)\mathcal{L}_2$ is appropriate.

**Example 1:** The case $l_0 = 3$ with $(d_2, d_3) = (1, 2)$. From (40) and (41) we have that $u_{2\,t} = 0$. Then if one takes

$$u(k, x, t) = k^2 + u_1(x, t)k + u_0(x, t), \qquad v(k, x, t) = v_1(x, t)k + v_0(x, t),$$

and sets

$$f_1(z^3) = \frac{1}{2}(1 + i\sqrt{3})z^3, \qquad f_2 \equiv 0,$$

in (47), one gets

$$\left\{\begin{array}{lcl} v_{1\,t} & = & -2\,u_{1\,x} + \dfrac{5}{9}\,v_1^2\,v_{1\,x}, \\[2ex] v_{0\,t} & = & \dfrac{1}{9}\left(-18\,u_{0\,x} + \,v_1^2\,v_{0\,x} + 4\,v_0\,v_1\,v_{1\,x}\right), \\[2ex] u_{1\,t} & = & \dfrac{1}{9}\left(v_1^2\,u_{1\,x} - 6\,v_0\,v_{1\,x} - 6\,v_1\,v_{0\,x} + 6\,v_1\,u_1\,v_{1\,x}\right), \\[2ex] u_{0\,t} & = & \dfrac{1}{9}\left(v_1^2\,u_{0\,x} - 6\,v_0\,v_{0\,x} + 6\,u_0\,v_1\,v_{1\,x}\right), \end{array}\right.$$

i.e., the dispersionless version of the *coupled Boussinesq system* (3.20b) in [16].

**Example 2:** The case $l_0 = 2$ with $(d_2, d_3) = (1, 1)$. Now, one can see that $v_{1\,t} = 0$. By setting

$$u(k, x, t) = u_1(x, t)k + u_0(x, t), \qquad v(k, x, t) = -k + v_0(x, t),$$

and

$$f_1(z^2) = -z^2, \qquad f_2 \equiv 0,$$

in (48), we find the system,

$$\begin{cases} v_{0\,t} &= -2\,u_{0\,x} - 2\,v_0\,u_{1\,x} - u_1\,v_{0\,x}, \\ \\ u_{1\,t} &= -4\,u_1\,u_{1\,x} + \dfrac{2}{3}\,v_{0\,x}, \\ \\ u_{0\,t} &= -u_1\,u_{0\,x} - 3\,u_0\,u_{1\,x} - \dfrac{2}{3}v_0\,v_{0\,x}\,. \end{cases}$$

This is the dispersionless version of the system (4.13) in [16].

## 3.4 Whitham deformations of cubic curves

There are four types of cubic curves of the form (15) given by the equations

| $M$ | 0 | 1 | 2 |
|---|---|---|---|
| | $k = p^3 + v_1\,p + v_0$ | $k = p^2 + v_0 + \frac{v_1}{p-w_1}$<br>$k = p + \frac{v_1}{p-w_1} + \frac{v_2}{(p-w_1)^2}$ | $k = p + \frac{v_{1,1}}{p-w_1} + \frac{v_{2,1}}{p-w_2}$ |

Note here that the Newton exponent $l_0$ is given by $l_0 = 3-M$. Also in [8], two cases in $M = 1$ are shown to be conformally equivalent, i.e. $p = \infty \leftrightarrow p = w_1$.

For $M = 0$ the Whitham deformations are reductions of our flows with $w \equiv 0$. But, in general, other Whitham deformations are not of that form. To illustrate this point let us take the class with $M = 1$ and $N = 2$. The corresponding Newton exponent is $l_0 = 2$ and there the branches of $\boldsymbol{p}$ have the following asymptotic behaviour as $z \to \infty$

$$\begin{cases} p_1(z) = z + \mathcal{O}\left(\frac{1}{z}\right), \\ p_2(z) = p_1(-z) = -z + \mathcal{O}\left(\frac{1}{z}\right), \\ p_3(z) = w_1 + \mathcal{O}\left(\frac{1}{z}\right). \end{cases}$$

Let us consider now the Whitham flows (17) associated with the puncture at $p = \infty$

$$\boldsymbol{Q}_{0,n} = (z^n)_{\oplus}(\boldsymbol{p}).$$

In terms of the potentials $\mathbf{u} = (w, v, u)^\top$ they read

$$\partial_t \mathbf{u} = J_0 \mathbf{a}, \tag{49}$$

where

$$\mathbf{a} = \left( V^{-1} \begin{pmatrix} (z^n)_\oplus(p_1) \\ (z^n)_\oplus(p_2) \\ (z^n)_\oplus(p_3) \end{pmatrix} \right)_+ .$$

One easily sees that all matrix elements of $V^{-1}$ are of order $\mathcal{O}\Big(\frac{1}{z}\Big)$ with the exception of

$$\Big(V^{-1}\Big)_{13} = 1 + \mathcal{O}\Big(\frac{1}{z}\Big).$$

On the other hand, we have

$$\begin{cases} (z^n)_\oplus(p_1) = & z^n + \mathcal{O}\Big(\frac{1}{z}\Big), \\ (z^n)_\oplus(p_2) = & (-z)^n + \mathcal{O}\Big(\frac{1}{z}\Big), \\ (z^n)_\oplus(p_3) = & (z^n)_\oplus(w_1) + \mathcal{O}\Big(\frac{1}{z}\Big). \end{cases}$$

Therefore one gets

$$\mathbf{a} = \left( z^n V^{-1} \begin{pmatrix} 1 \\ (-1)^n \\ 0 \end{pmatrix} \right)_+ + (z^n)_\oplus(w_1)\, \mathbf{e}_3,$$

where $\mathbf{e}_3 = (0, 0, 1)^\top$, so that equation (49) becomes

$$\partial_t \mathbf{u} = J_0 \Big( T\, \nabla_{\mathbf{u}} [z^n (p_1 + (-1)^n p_2)] \Big)_+ + J_0 \Big( (z^n)_\oplus(w_1)\, \mathbf{e}_3 \Big). \tag{50}$$

Similar expressions can be obtained for the deformations generated by the Whitham flows (17) for $\alpha = 1$ and $n \geq 1$.

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## References


[1] S. P. Novikov, S. V. Manakov, L. P. Pitaevski and V. E. Zakharov *Theory of solitons. The inverse scattering method*, Plenum, New York (1984).

[2] E. D. Belokolos, A. I. Bobenko , V. Z. Enolski, A. R. Its and V. B. Matveev, *Algebro-Geometric approach to nonlinear integrable equations*, Springer-Verlag, Berlin (1994).

[3] B. Dubrovin and S. Novikov, Russ. Math. Surv. **44**(6), 35 (1989).

[4] H. Flaschka, M.G. Forest and D.W. Mclauglin , Commun. Pure Appl. Math **33**, 739 (1980).

[5] B.A. Dubrovin, Commun. Math. Phys. **145**, 415 (1992)

[6] I.M. Krichever, Funct. Anal. Appl. **22**, 206 (1988).

[7] I.M. Krichever, Commun. Pure. Appl. Math. **47**, 437 (1994).

[8] A. Aoyama and Y. Kodama, Commun. Math: Phys. **182**, 185 (1996)

[9] Y. Kodama and B.G. Konopelchenko, J. Phys. A: Math. Gen. **35**, L489-L500 (2002); *Deformations of plane algebraic curves and integrable systems of hydrodynamic type* in *Nonlinear Physics: Theory and Experiment II*,edited by M.J. Ablowitz et al, World Scientific, Singapore (2003).

[10] B.G. Konopelchenko and L. Martínez Alonso, J. Phys. A: Math. Gen. **37**, 7859 (2004).

[11] B. L. van der Waerden,*Algebra*, Vol. I, Springer-Verlag, Berlin (1991).

[12] L. Redei,*Introduction to algebra*, Vol. I, Pergamon Press, Oxford (1967).

[13] I.G.Macdonald, *Symmetric functions and Hall polynomials*, Clarendom Press, Oxford (1979).

[14] R. Y. Walker, *Algebraic Curves*, Springer-Verlag, Berlin (1978).

[15] S. S. Abhyankar, *Algebraic Geometry for Scientists and Engineers*, Mathematical Surveys and Monograps vol. 35, AMS (1990).

[16] M. Antonowicz, A. P. Fordy and Q. P. Liu, *Nonlinearity* **4**, 669 (1991)